Different Strokes in Randomised Strategies: Revisiting Kuhn's Theorem Under Finite-memory Assumptions

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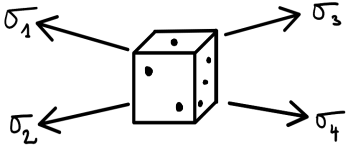
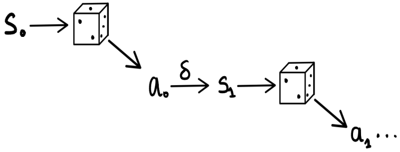
Introduction

- In general, one can define randomised strategies in different ways.



Mixed strategies



Behavioural strategies

- In general, these two classes of strategies are not comparable.

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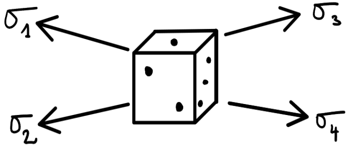

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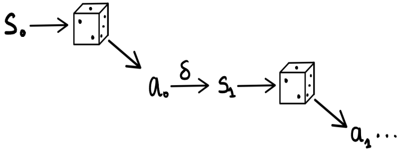

Behavioural strategies

- In general, these two classes of strategies are not comparable.
- Kuhn's theorem [Aum64][1]: in games of perfect recall any mixed strategy has an equivalent behavioural strategy and vice-versa.

Focus of this talk

A Kuhn-inspired classification of finite-memory strategies.

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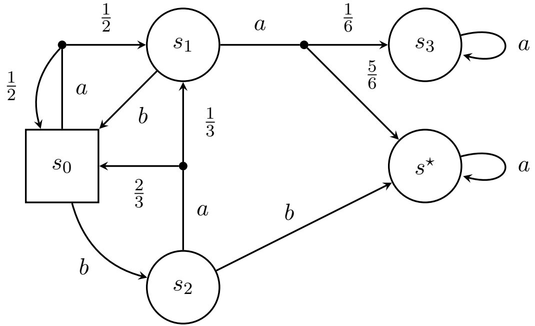
Setting: finite stochastic games

We consider two-player stochastic games.



Essential characteristics

- Finite state space $S = S_1 \uplus S_2$ and action space A.
- Probabilistic transition function $\delta\colon S \times A \to \mathcal{D}(S)$.
- No deadlocks.

Strategies

Definition

A strategy of $\mathcal{P}_i$ is a function $\sigma_i \colon (SA)^* S_i \to \mathcal{D}(A)$.

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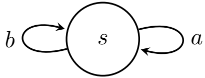
- Equality is too restrictive: two different strategies may induce the same behaviour in practice.



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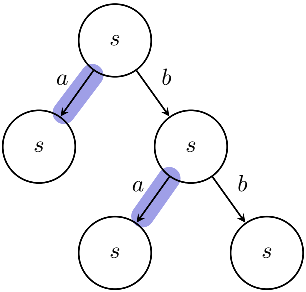

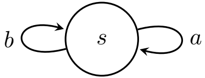

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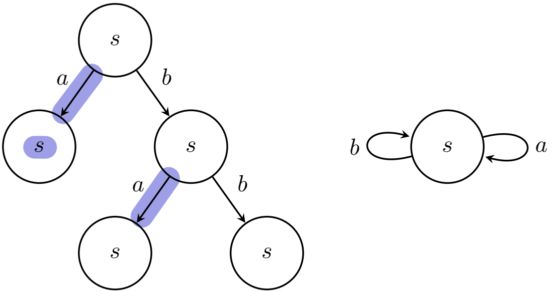

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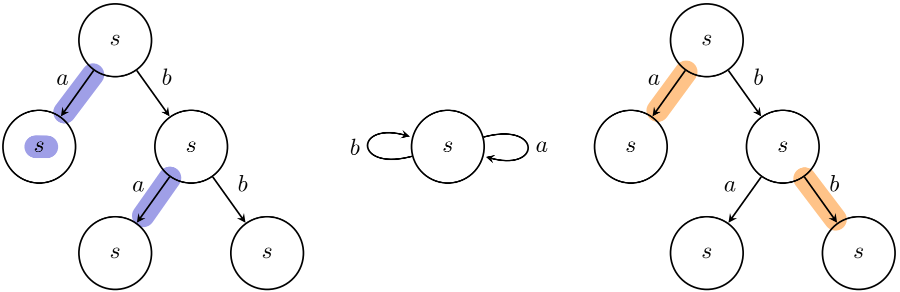

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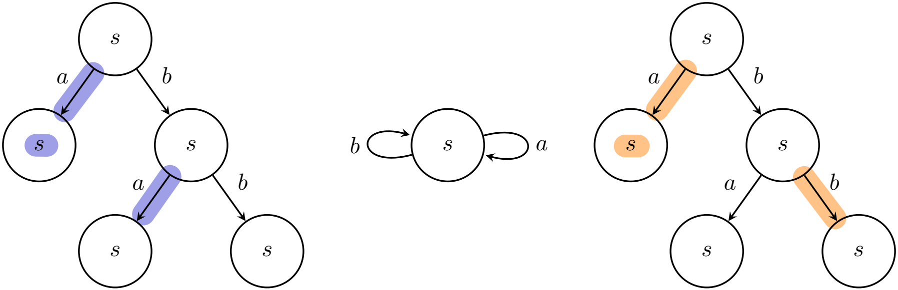

Outcome-equivalence

Given two strategies σ_1 and σ_2, and an initial state $s_{\mathsf{init}} \in S$, we define a probability distribution on the set of plays in the usual way: for any history $h = s_0 a_0 s_1 \dots s_n$ with $s_0 = s_{\mathsf{init}}$, we set

$$\mathbb{P}_s^{\sigma_1,\sigma_2}(\mathsf{Cyl}(h)) = \prod_{k=0}^{n-1} \sigma_{i(k)}(s_0 a_0 \dots s_k) \cdot \delta(s_k, a_k, s_{k+1})$$

where $\mathsf{Cyl}(h)$ is the set of plays with h as a prefix, and $i(k) = 1$ if $s_k \in S_1$ and 2 otherwise.

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Outcome-equivalence

Two strategies σ_1 and τ_1 of $\mathcal{P}_1$ are outcome-equivalent if for all strategies σ_2 of $\mathcal{P}_2$ and all initial states $s_{\mathsf{init}} \in S$, we have

$$\mathbb{P}_{s_{\mathsf{init}}}^{\sigma_1,\sigma_2} = \mathbb{P}_{s_{\mathsf{init}}}^{\tau_1,\sigma_2}.$$

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Randomised finite-memory strategies

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A strategy σ_i of $\mathcal{P}_i$ is finite-memory if it is induced by a stochastic Mealy machine $\mathcal{M} = (M, \mu_{\mathsf{init}}, \alpha_{\mathsf{next}}, \alpha_{\mathsf{up}})$ where

- M is a finite set of memory states;
- $\mu_{\mathsf{init}} \in \mathcal{D}(M)$ is an initial distribution;
- $\alpha_{\mathsf{next}} \colon M \times S_i \to \mathcal{D}(A)$ is a stochastic next-move function;
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- We can classify Mealy machines following whether their initialisation, updates and outputs are randomised or deterministic.

All classes of Mealy machines are not equally powerful

- Some classes of Mealy machines allow richer behaviours than others.

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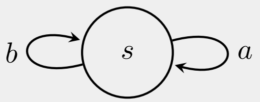
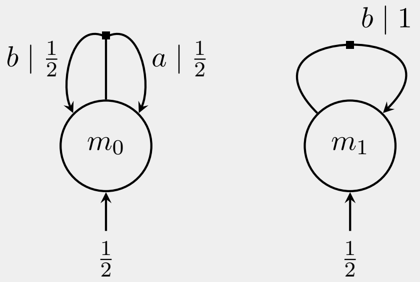
- Some classes of Mealy machines allow richer behaviours than others.
- For instance, the strategy illustrated on the right cannot be emulated with randomisation only in the outputs.

Game



Mealy machine example



Our results

We use acronyms to define classes of Mealy machines: we use XYZ where X, Y, Z $\in$ {D, R} where D stands for deterministic and R for random, and

- X characterises initialisation,
- Y characterises outputs (next-move function),
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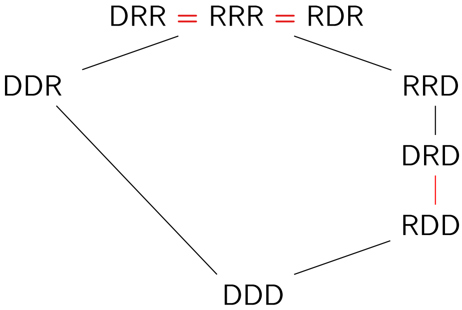
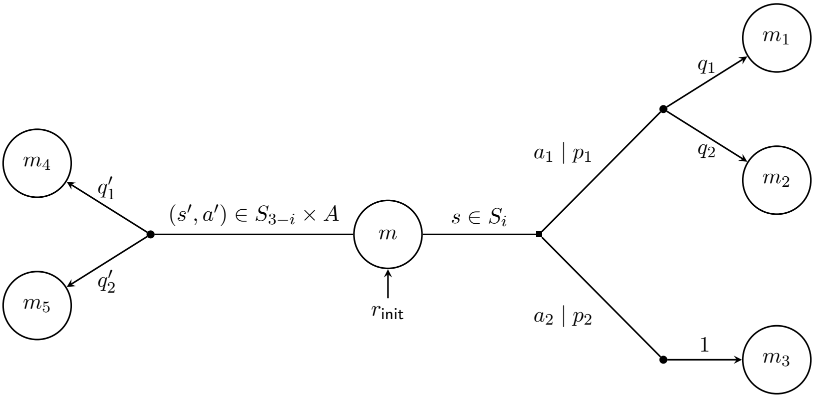


Illustrating a finite-memory strategy

In the sequel, we will illustrate fragments of Mealy machines for $\mathcal{P}_i$ as follows.



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RDD $\subseteq$ DRD: trading random initialisation for outputs

We fix an RDD Mealy machine $\mathcal{M} = (M, \mu_{\text{init}}, \alpha_{\text{next}}, \alpha_{\text{up}})$.

- We use an adaptation of the subset construction to go from $\mathcal{M}$ to a DRD Mealy machine.
- State space of functions $f\colon \text{supp}(\mu_{\text{init}}) \to (M \cup \{\bot\})$:
  - We simulate the strategy from each initial state.
  - If an action is inconsistent with one of the simulations, we stop it (symbolised by $\bot$).

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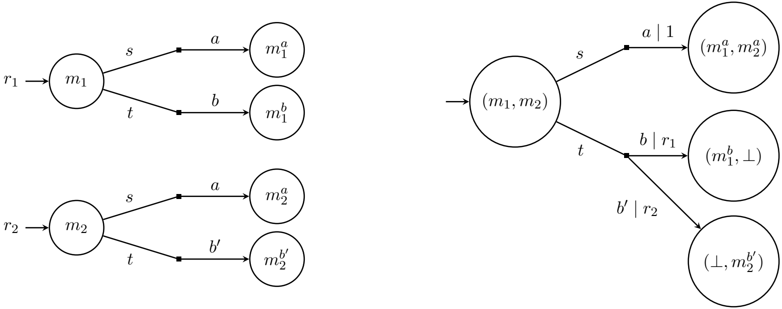

RRR $\subseteq$ DRR: determinising initialisation

We fix an RRR Mealy machine $\mathcal{M} = (M, \mu_{\mathsf{init}}, \alpha_{\mathsf{next}}, \alpha_{\mathsf{up}})$.

- To derive a DRR Mealy machine from $\mathcal{M}$, we add a new initial state m_{new} to the memory state space.
- We use stochastic updates to return to $\mathcal{M}$ from m_{new}. Transition probabilities are chosen so the distribution over memory states is the same in $\mathcal{M}$ and the DRR Mealy machine after the first step.

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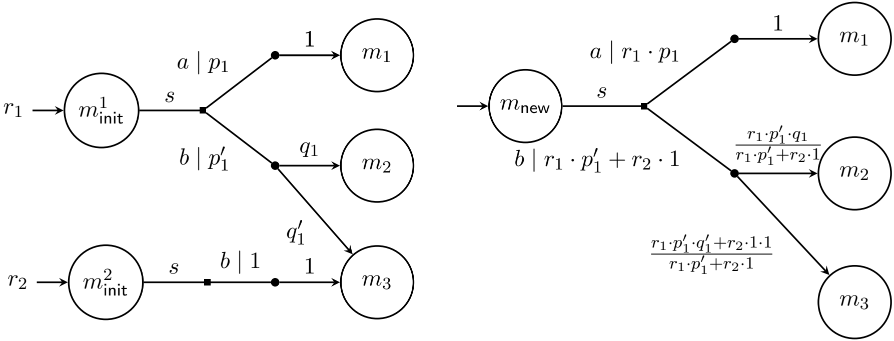

RRR $\subseteq$ RDR: determinising outputs

We fix an RRR Mealy machine $\mathcal{M} = (M, \mu_{\mathsf{init}}, \alpha_{\mathsf{next}}, \alpha_{\mathsf{up}})$.

- To derive a RDR Mealy machine from $\mathcal{M}$, we expand the state space by augmenting memory states with pure memoryless strategies $\sigma_i \colon S_i \to A$.
- We use stochastic initialisation and updates to integrate the randomisation over actions in the transitions.

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Naive construction $\rightsquigarrow$ memory state space grows by a factor of $|A|^{|S_i|}$

$\hookrightarrow$ We can do better:

Theorem

There exists an RDR Mealy machine with $|M| \cdot |S_i| \cdot |A|$ states whose induced strategy is outcome-equivalent to $\mathcal{M}$.

RRR $\subseteq$ RDR: choosing pure memoryless strategies

- Consider a game such that $S_i = \{s_1, s_2, s_3\}$, and $A = \{a_1, a_2, a_3\}$. Assume that for a memory state $m \in M$, we have:
  - $\alpha_{\mathsf{next}}(m, s_1)(a_1) = \alpha_{\mathsf{next}}(m, s_1)(a_2) = \frac{1}{2}$;
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  - $\alpha_{\mathsf{next}}(m, s_3)(a_1) = \frac{1}{3}$, $\alpha_{\mathsf{next}}(m, s_3)(a_2) = \frac{1}{6}$ and $\alpha_{\mathsf{next}}(m, s_3)(a_3) = \frac{1}{2}$.

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- We represent the actions in a table to derive the pure memoryless strategies and their probabilities.

<table>
<tr><td>s_1</td><td colspan="2">a_1</td><td colspan="2">a_2</td></tr>
<tr><td>s_2</td><td>a_1</td><td colspan="2">a_2</td><td>a_3</td></tr>
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s_1	a_1		a_2	
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s_3	a_1	a_2	a_3	

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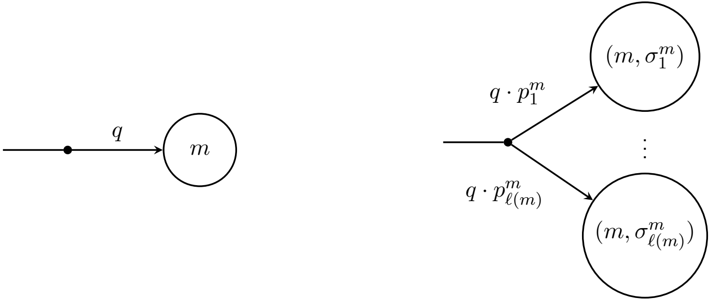
<tr><td>σ_k</td><td>σ_1</td><td>σ_2</td><td>σ_3</td><td>σ_4</td></tr>
</table>

$x_1 = 0$ $x_2 = \frac{1}{3}$ $x_3 = \frac{1}{2}$ $x_4 = \frac{2}{3}$ $x_5 = 1$

RRR $\subseteq$ RDR: exploiting the memoryless strategies

- For each memory state $m \in M$, we determine pure memoryless strategies $\sigma_1^m, \ldots, \sigma_{\ell(m)}^m$ and their respective probabilities $p_1^m, \ldots, p_{\ell(m)}^m$.
- We split transitions that enter m into transitions that go to the states (m, σ_j^m): a transition of probability q into m yields a transition with probability $q \cdot p_j^m$ into (m, σ_j^m).



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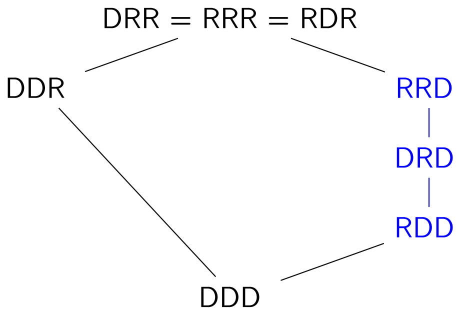
Differences between classes

We discuss the following aspects:

- The chain of inclusions DDD $\subsetneq$ RDD $\subsetneq$ DRD $\subsetneq$ RRD $\subsetneq$ RRR is strict.
- It holds that DDR $\not\subseteq$ RRD and RDD $\not\subseteq$ DDR.



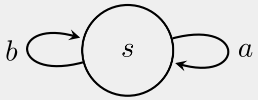
Strictness: $\mathrm{RDD} \subsetneq \mathrm{DRD}$

- In a one-player deterministic game, RDD strategies have finitely many outcomes.

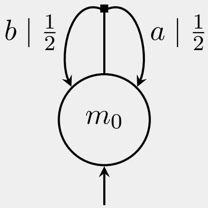
Strictness: $\mathrm{RDD} \subsetneq \mathrm{DRD}$

- In a one-player deterministic game, RDD strategies have finitely many outcomes.
- The DRD strategy depicted below has no RDD equivalent.

Game



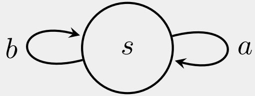
Witness of $\mathrm{RDD} \subsetneq \mathrm{DRD}$



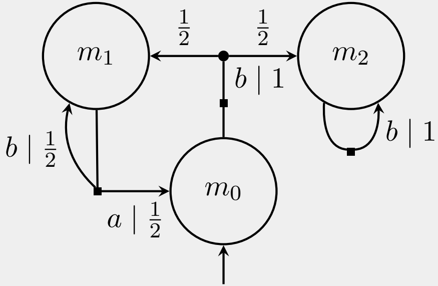
Strictness: $\mathrm{DDR} \not\subseteq \mathrm{RRD}$

- The number of memory states in which we can find ourselves as a play goes on cannot increase for an RRD strategy..
- To have a positive probability of never using a, we must eventually be in a memory state m such that $\alpha_{\mathsf{next}}(m, s)(a) = 0$ with positive probability.

Game



Can be adapted to witness $\mathrm{DDR} \not\subseteq \mathrm{RRD}$



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Taxonomy in settings of partial information

- Up to now, we have discussed a classification of strategies in a setting of perfect information.

$\rightsquigarrow$ Can we weaken this hypothesis ?

- It is not necessary to see the states themselves.
- For the inclusion RDD $\subseteq$ DRD, we rely on the visibility of actions in our subset construction.
- For the inclusion RRR $\subseteq$ DRR, we also use the visibility of actions in conditional probabilities.

Partial information

The classification holds in games where $\mathcal{P}_i$ can see their actions and distinguish the owner of states from their observations.

References I

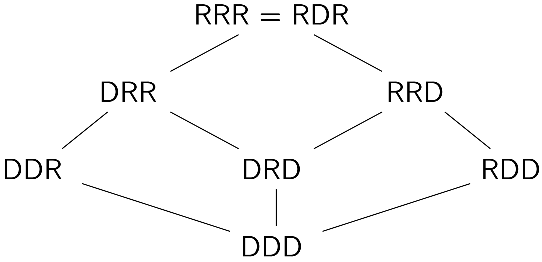
Aumann, Robert J . "28. Mixed and Behavior Strategies in Infinite Extensive Games". In: *Advances in Game Theory. (AM-52), Volume 52*. Princeton University Press, 2016, pp. 627–650. DOI: doi:10.1515/9781400882014-029. URL: https://doi.org/10.1515/9781400882014-029.

Collapses – invisible actions

What happens to the lattice in full generality ? If we assume nothing on the visibility of actions ?

- Two inclusions of our lattice no longer hold. We have:
  - RDD$\not\subseteq$DRD;
  - RRR$\not\subseteq$DRR (we even have RDD$\not\subseteq$DRR).
- Intuitively, for a strategy with deterministic outputs (i.e., in a subclass of RDR), the output actions are encoded in the Mealy machine itself. $\rightsquigarrow$ such strategies allow the same behaviours whether actions are visible or not.

General lattice: no hypotheses on actions



Subgame perfect equilibria and Kuhn's theorem

- In the statement of Kuhn's theorem and our classification, the output of the strategies along inconsistent branches histories are completely disregarded.
- In other words, our classification approach is not relevant for the study of subgame perfect equilibria, for which these inconsistent histories are nonetheless taken in account.
- However, the output of a finite-memory strategy along an inconsistent history is not well-defined.